A note on generalized stationary iterative method for solving saddle point problems

Li-Tao Zhang[1,*], Hui-Li Meng[2]

[1]*Department of Mathematics and Physics, Zhengzhou Institute of Aeronautical Industry Management, Zhengzhou, Henan, 450015, P. R. China*

[2]*College of Computer and Information Engineering, Henan Normal University, Xinxiang, 453007, P.R. China*

Received 20 October 2014, www.cmnt.lv

Abstract

Recently, Miao and Wang [Journal of Applied Mathematics and Computing, 35(2011):459-468] studied the convergence of the generalized stationary iterative (GSI) method for solving the saddle point problems. In this paper, based on Miao and Wang's convergence theorem, we perfect it and give new convergence conditions. Moveover, by using relaxation technique, we present an improved generalized stationary iterative (IGSI) method for solving the saddlepoint problems and analyze the convergence of the corresponding method.

Keywords: Saddle point problem, Generalized stationary iterative method, Convergence. MSC: 65F10; 65F15; 65F50

1 Introduction

Consider the saddle point problems of the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1}$$

where $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ be of full column rank and $b \in R^m$ and $q \in R^n$ are given vectors, with $m \geq n$.

Systems of the form (1) appears in many different applications of scientific computing, such as constrained optimization [1], the finite element method for solving the Navier-Stokes equation [2-4], and constrained least squares problems and generalized least squares problems [5-8]. There have been several recent papers [9-37] for solving the augmented system (1). Santos et al. [6] studied preconditioned iterative methods for solving the augmented system (1) with A = I. Yuan et al. [7, 8] proposed several variants of SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from generalized least squares problems where A can be symmetric and positive Semide finite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [20] presented SOR-like algorithms for solving system (1). Darvishi et al. [19] studied SSOR method for solving the augmented systems. Bai et al. [9, 10, 18, 36] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving systems (1). Zhang and Lu [27] showed the generalized symmetric SOR method for augmented systems. Peng and Li [23] studied unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [11-15, 24] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's

2009 LAA paper [24] and Chen and Jiang's 2008 AMC paper [18] studied some general approaches about the relaxed splitting iteration methods. Wu, Huang and Zhao [25] presented modified SSOR (MSSOR) method for augmented systems (1). Zhang et al. [28, 29] established a generalized MSSOR (GMSSOR) method for augmented systems and analyze convergence of the corresponding method. Recently, Miao et al. [22] studied the convergence of the generalized stationary iterative (GSI) method.

In this paper, we establish an improved generalized stationary iterative (IGSI) method for solving the saddle point problems and analyze convergence of the corresponding method. Moreover, based on Miao and Wang's convergence theorem [22], we perfect it and give new convergence conditions.

* Corresponding author's E-mail: litaozhang@163.com

2 Improved GSI method

For the sake of simplicity, Golub et al. [20] rewrite system (1) as:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b \\ -q \end{pmatrix}. \quad (2)$$

Recently, for the coefficient matrix of the augmented system (1), Miao et al. [22] make the following splitting:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}=\begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \alpha Q \end{pmatrix}-\begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \alpha Q \end{pmatrix}, \quad (3)$$

where α and β are real parameters with $\alpha \neq 0$, Q is a nonsingular matrix.

Based the above splitting, by using relaxation technique, we propose the following splitting:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}=\begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \gamma Q \end{pmatrix}-\begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \gamma Q \end{pmatrix}, \quad (4)$$

where α, β and γ are real parameters with $\alpha \neq 0$, $\gamma < 0$, Q is a nonsingular matrix. Then we can obtain the following improved generalized stationary iterative (IGSI) scheme:

$$\begin{pmatrix} x^{(k+1)} \\ y^{k+1} \end{pmatrix}=H\begin{pmatrix} x^{(k)} \\ y^{k} \end{pmatrix}+\begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \gamma Q \end{pmatrix}^{-1}\begin{pmatrix} b \\ -q \end{pmatrix}, k=0,1,2,\dots$$

where

$$H=\begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \gamma Q \end{pmatrix}^{-1}\begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \gamma Q \end{pmatrix} \quad (5)$$

is the IGSI iterative matrix.

Improved GSI method: Let $Q \in R^{n\times n}$ be a nonsingular and symmetric matrix. Given initial vectors $x^{(0)} \in R^m$ and $x^{(0)} \in R^n$ and three relaxed parameters $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ .For k = 0,1,2,... until the iteration sequence $\{((x^k)^T,(y^k)^T)^T\}$ converges, compute

$$\begin{cases} x^{(k+1)}=(1-\frac{1}{\alpha})x^{(k)}+\frac{1}{\alpha}A^{-1}(b-By^{(k)}), \\ y^{(k+1)}=y^{(k)}+\frac{1}{\gamma}Q^{-1}\{B^T\left(\beta x^{(k+1)}+(1-\beta x^{(k)})-q\right)\} \end{cases}$$

and Q is an approximate (preconditioning) matrix of the Schur complement matrix $B^T A^{-1} B$.

Remark 2.1 When the relaxed parameters $\gamma=\alpha$, IGSI method reduces to GSI method; When $\gamma=\alpha=\frac{1}{\omega}$ and $\beta=1$; IGSI method reduces to SOR-like method [16]. When $\gamma=\alpha=\frac{1}{\omega}$ and $\beta=\frac{\gamma}{\omega}$, IGSI-like method reduces to GAOR method [21]. So, IGSI method is the generalization of these methods. Furthermore, IGSI method with appropriate parameters will have better convergence rate.

3 Convergence of IGSI method

Lemma 3.1 [26] Consider the quadratic equation $x^2-bx+c=0$,where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c|<1$ and $|b|<1+c$.

Lemma 3.2 [22] Let $H_{\alpha,\beta}$ be the iteration matrix of GSI method . If $m>n$,then

(i) $\lambda=1-\frac{1}{\alpha}$ is an eigenvalue of $H_{\alpha,\beta}$ at least with multiplicity of m − n;

(ii) The other eigenvalues λ of $H_{\alpha,\beta}$ determined by the functional equation

$$(1-\beta+\lambda\beta)\mu=(\lambda\alpha+1-\alpha)(\lambda\alpha-\alpha), \quad (6)$$

where

$$H_{\alpha,\beta}=\begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \gamma Q \end{pmatrix}-\begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \alpha Q \end{pmatrix}$$

Recently, based the splitting scheme (3) and the equation (6), Miao et al. [22] gave the following convergence Theorem:

Theorem 3.3 [22] Let A and Q be symmetric positive definite, and B be of full column rank. Suppose that all eigenvalues μ of $Q^{-1}B^T A^{-1}B$ are real and positive. Denote the largest eigenvalues of the matrix μ of $Q^{-1}B^T A^{-1}B$ by $\mu_{\max}$. Then the GSI method is convergence if α and β satisfy

$$\alpha > \max\{\frac{1}{2},\frac{\sqrt{\mu_{\max}}}{2}\},$$

and

$$1-\frac{\alpha}{\mu_{\max}}<\beta<\frac{1}{2}+\frac{\alpha(2\alpha-1)}{\mu_{\max}}. \quad (7)$$

Through analyzing the proving process of Theorem 3.3, we further perfect it and give new convergence conditions, which is as follows.

Theorem 3.4 Let A and Q be symmetric positive definite and symmetric, respectively, and B be of full column rank. Suppose that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real and negative. Denote the largest eigenvalues of the matrix μ of $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$: Then the GSI method is convergence if
and satisfy

$$\alpha > \max\{\frac{1}{2}, \frac{\sqrt{-\mu_{\max}}}{2}\},$$

and

$$\frac{1}{2} - \frac{\alpha(2\alpha-1)}{\mu_{\max}} < \beta < 1 + \frac{\alpha}{\mu_{\max}}, \qquad (8)$$

Proof Let $\lambda \in \sigma(H_{\alpha,\beta})$ and $\lambda \neq 0$. Then $\lambda = 1 - \frac{1}{\alpha}$ or from (6) λ satisfies

$$\alpha^2\lambda^2 - (2\alpha^2 - \alpha + \beta\mu)\lambda + \alpha^2 - \alpha + (\beta-1)\mu = 0$$

By Lemma 3.1, $|\lambda| < 1$ if and only if

$$\left|1 - \frac{1}{\alpha}\right| < 1, \qquad (9)$$

and

$$\begin{cases} \left|\frac{2\alpha^2 - \alpha + \beta\mu}{\alpha^2}\right| = 1 + \frac{\alpha^2 - \alpha + (\beta-1)\mu}{\alpha^2}, \\ \left|\frac{\alpha^2 - \alpha + (\beta-1)\mu}{\alpha^2}\right| < 1 \end{cases}, \qquad (10)$$

From (10), we can obtain

$$\begin{cases} \alpha > (\beta-1)\mu \\ 2\alpha^2 - \alpha + (\beta-1)\mu > 0 \end{cases} \qquad (11)$$

and

$$\begin{cases} 4\alpha^2 - 2\alpha + 2\beta\mu - \mu > 0 \\ \mu < 0 \end{cases} \qquad (12)$$

In terms of (11) and (12), and note that (9) is valid, then

$$\alpha > \frac{1}{2},$$

and

$$\frac{1}{2} - \frac{\alpha(2\alpha-1)}{\mu_{\max}} < \beta < 1 + \frac{\alpha}{\mu_{\max}}.$$

Meanwhile, in order that there is an satisfying (8), $\beta < 1 + \frac{\alpha}{\mu_{\max}}$ should be great than $\frac{1}{2} - \frac{\alpha(2\alpha-1)}{\mu_{\max}}$ or equality, $\alpha > \frac{\sqrt{-\mu_{\max}}}{2}$; thus should be $\alpha > \max\{\frac{1}{2}, \frac{\sqrt{-\mu_{\max}}}{2}\}$.

Based on the IGSI method and using the similar proving process of Theorem 3.4 [22], we give the following convergence Theorem. Moreover, we may ensure that Q is symmetric positive definite.

Theorem 3.5 Let H be the iteration matrix of IGSI method. If m > n; then

(i) $\lambda = 1 - \frac{1}{\alpha}$ is an eigenvalue of H at least with multiplicity of m − n;

(ii) The other eigenvalues H determined by the functional equation

$$(1 - \beta + \lambda\beta)\mu = (\lambda\alpha + 1 - \alpha)(\lambda\gamma - \gamma), \qquad (13)$$

where

$$H = \begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \gamma Q \end{pmatrix}^{-1} \begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \gamma Q \end{pmatrix}.$$

Proof Using the similar proving process, we easily get the above conclusion.

Theorem 3.6 Let A and Q be symmetric positive definite, and B be of full column rank. Suppose that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real and positive. Denote the largest eigenvalues of the matrix μ of $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$: Then the IGSI method is convergence if the parameters satisfy

$$\alpha > \max\{\frac{1}{2}, \frac{\mu_{\max} + \gamma}{1 - 4\gamma}\},$$

and

$$1 + \frac{\gamma}{\mu_{\max}} < \beta < \frac{1}{2} + \frac{\alpha + \gamma - 4\alpha\gamma}{2\mu_{\max}}, \qquad (14)$$

Proof Let $\lambda \in H$ and $\lambda \neq 0$, Then $\lambda = 1 - \frac{1}{\alpha}$, or from (13) λ satisfies

$$\alpha^2\lambda^2 - (2\lambda\alpha - \alpha + \beta\mu)\lambda + \alpha\lambda - \lambda + (\beta-1)\mu = 0$$

By Lemma 3.1, $|\lambda| < 1$ if and only if

$$\left|1 - \frac{1}{\alpha}\right| < 1, \qquad (15)$$

and

$$\begin{cases} \left|\frac{2\lambda\alpha - \alpha + \beta\mu}{\alpha\gamma}\right| = 1 + \frac{\alpha\gamma - \gamma + (\beta-1)}{\alpha\gamma} \\ \left|\frac{\alpha\gamma - \gamma + (\beta-1)}{\alpha\gamma}\right| < 1 \end{cases}, \qquad (16)$$

From (16), we can obtain

$$\begin{cases} \lambda < (\beta-1)\mu \\ 2\alpha\gamma - \gamma + (\beta-1) > 0 \end{cases}, \qquad (17)$$

and

$$\begin{cases} 0 < \alpha - \lambda < \mu \\ 4\alpha\gamma - \alpha - \gamma + 2\beta\mu - \mu < 0 \end{cases} \qquad (18)$$

In terms of (17) and (18), and note that (15) is valid, then

$\alpha > \frac{1}{2}$,

and

$1 + \frac{\gamma}{\mu_{\max}} < \beta < \frac{1}{2} + \frac{\alpha + \gamma - 4\gamma\alpha}{2\mu_{\max}}$.

Meanwhile, in order that there is an satisfying (14), $\frac{1}{2} + \frac{\alpha + \gamma - 4\gamma\alpha}{2\mu_{\max}}$ should be great than $1 + \frac{\gamma}{\mu_{\max}}$; or equality, $\alpha > \frac{\mu_{\max} + \gamma}{1 - 4\gamma}$; thus α should be $\alpha > \max\{\frac{1}{2}, \frac{\mu_{\max} + \gamma}{1 - 4\gamma}\}$.

Acknowledgments

This research of this author is supported by NSFC Tianyuan Mathematics Youth Fund (11226337), NSFC(61203179, 61202098, 61170309, 91130024, 61033009, 61272544,61472462 and 11171039), Aeronautical Science Foundation of China (2013ZD55006), Project of Youth Backbone Teachers of Colleges and Universities in Henan Province(2013GGJS-142), ZZIA Innovation team fund (2014TD02), Major project of development foundation of science and technology of CAEP (2012A0202008), Defense Industrial Technology Development Program, Basic and Advanced Technological Research Project of of Henan Province (122300410181, 132300410373, 142300410333), Natural Science Foundation of Henan Province (13A110399,14A630019,14B110023), Natural Science Foundation of Zhengzhou City (141PQYJS560).

References

[1] Wright S 1997 Stability of augmented system factorizations in interior-point methods *SIAM J. Matrix Anal. Appl.* **18**, 191-222

[2] Elman H, Silvester D 1996 Fast nonsymmetric iterations and preconditioning for NavierCStokes equations *SIAM J. Sci. Comput.* **17**, 33-46

[3] Elman H, Golub G H 1994 Inexact and preconditioned Uzawa algorithms for saddle point problems *SIAM J. Numer. Anal.*, **31**, 1645-61

[4] Fischer B, Ramage A, Silvester D J, Wathen A J 1998 Minimum residual methods for augmented systems *BIT* **38**, 527-43

[5] Arioli M, Duff I S and de Rijk P P M 1989 On the augmented system approach to sparse least-squares problems *Numer. Math.,* , **55**, 667-84

[6] Santos C H, Silva B P B, Yuan J-Y 1998 Block SOR methods for rank deficient least squares problems *J. Comput. Appl. Math.,* , **100**, 1-9

[7] Yuan J-Y 1996 Numerical methods for generalized least squares problems *J. Comput. Appl. Math.* **66**, 571-84

[8] Yuan J-Y, Iusem, A N 1996 Preconditioned conjugate gradient method for generalized least squares problems *J. Comput. Appl. Math.* **71**, 287-97

[9] Bai Z-Z, Parlett B N, Wang Z. Q 2005 On generalized successive over relaxation methods for augmented linear systems *Numer. Math.* **102**, 1-38

[10] Bai Z-Z, Wang Z-Q 2008 On parameterized inexact Uzawa methods for generalized saddle point problems *Linear Algebra Appl.* **428**, 2900-32

[11] Bai Z-Z, Yang X 2009 On HSS-based iteration methods for weakly nonlinear systems *Appl. Numer. Math.* **59** 2923-36

[12] Bai Z-Z, Golub G H, Michael K N 2008 On inexact hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems *Linear Algebra Appl.* **428**, 413-40

[13] Bai Z-Z 2008 Several splittings for non-Hermitian linear systems, *Science in China, Series A*: *Math.* **51** 1339-48

[14] Bai Z-Z, Golub G H, Lu L-Z, Yin J-F 2005 Block-Triangular and skew-Hermitian splitting methods for positive definite linear systems *SIAM J. Sci. Comput* **26** 844-63

[15] Bai Z-Z, Golub G H, Ng M K Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems *SIAM J. Matrix. Anal.* A **24** 603-26

[16] Bai Z-Z, Golub G H, Ng M K On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iteration. Available online at http://www.sccm.stanford.edu/wrap/pub-tech.html.

[17] Bai Z-Z, Golub G H, Pan J –Y 2002 Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems. *Technical Report SCCM-02-12, Scientic Computing and Computational Mathematics Program, Department of Computer Science* Stanford University, Stanford, CA

[18] Chen F, Jiang Y-L 2008 A generalization of the inexact parameterized Uzawa methods for saddle point problems *Appl. Math. Comput* **206** 765-71

[19] Darvishi M T, Hessari P 2006 Symmetric SOR method for augmented systems *Appl. Math. Comput,* **183** 409-15

[20] Golub G. H., Wu X, Yuan J-Y 2001 SOR-like methods for augmented systems *BIT* **41** 71-85

[21] Li C.-J., Li Z., Nie Y.-Y., Evans D. J., 2004 Generalized AOR method for the augmented system *Int. J. Comput. Math*. **81** 495-504

[22] Miao S-X and Wang K 2011 On generalized stationary iterative method for solving the saddle point problems *J. Appl. Math. Comput.* **35** 459-68

[23] Peng X-F and Li W 2008 On unsymmetric block overrelaxation-type methods for saddle point, *Appl. Math. Comput*. **203**(2) 660-71

[24] Wang L, Bai Z-Z 2008 Convergence conditions for splitting iteration methods for non-Hermitian linear systems *Linear Algebra Appl.* **428** 453-68

[25] Wu S-L, Huang T-Z and Zhao X-L 2009 A modified SSOR iterative method for augmented systems *J. Comput. Appl. Math.*, **228**(1) 424-33

[26] Young D M 1971 *Iteration Solution for Large Systems* New York: Academic Press

[27] Zhang G-F, Lu Q-H 2008 On generalized symmetric SOR method for augmented systems J. *Comput. Appl. Math.* **1**(15) 51-8

[28] Zhang L-T, Huang T-Z, Cheng S-H, Wang Y-P 2012 Convergence of a generalized MSSOR method for augmented systems *J. Comput. Appl. Math.* **236** 1841-50

[29] Zhang L-T 2014 A new preconditioner for generalized saddle matrices with highly singular (1,1) blocks *Int. J. Comput. Math.* http://dx.doi.org/10.1080/00207160.2013.867953.

[30] Zhang L-T, Li J-L 2014 The weaker convergence of modulus-based synchronous multisplitting multi-parameters methods for linear complementarity problems *Comput. Math. Appl.* **67** 1954-9

[31] Zhang L-T, Huang T-Z, Gu T-X 2008 Global relaxed non-stationary multisplitting multiparameters methods *Int. J. Comput. Math.* **85** 211-24

[32] Zhang L-T, Huang T-Z, Gu T-X 2008 Convergence of relaxed multisplitting USAOR method for an H-matrix *Appl. Math. Comput,* **202** 121-32

[33] Zhang L-T, Huang T-Z, Gu T-X 2009 Convergent improvement of SSOR multisplitting method *J. Comput. Appl. Math.* **225** 393-7

[34] Zhang L-T, Huang T-., Cheng S-H, Gu T-., Wang Y-P 2011 A note on parallel multisplitting TOR method of an H-matrix *Int. J. Comput. Math.* **88** 501-7

[35] Zhang L-T, Huang T-Z, Cheng S-H, Gu T-X. 2011 The weaker convergence of non-stationary matrix multisplitting methods for almost linear system *Taiwan. J. Math.* **15** 1423-36

[36] Zhang L-T, Zuo X-Y, Gu T-X, Liu X.P 2014 Improved convergence theorems of multisplitting methods for the linear complementarity problem *Appl. Math. Comput* Accepted, 10.1016/j.amc.2014.06.038.

[37] Zheng B, Bai Z-Z, Yang X 2009 On semi-convergence of parameterized Uzawa methods for singular saddle point problems, *Linear Algebra Appl.*431 808-17

Authors

Zhang Litao, born in March, 1980, Zhengzhou city, P.R. China

Current position, grades: Associate Professor China of Zhengzhou Institute of Aeronautical Industry Management.
University studies: B.Sc. of School of Mathematical Sciences from University of Electronic Science and Technology of China in China. M.Sc. of College of Mathematics and Information Science from Henan Normal University in China.
Scientific interest: Parallel Computing, Preconditioning, Iterative methods, Linear systems
Publications: more than 50 papers published in various journals.
Experience: teaching experience of 5 years, 6 scientific research projects.

Meng Huili, born in January, 1978, Xinxiang city, P.R. China

Current position, grades: Lecturer China of Henan Normal University.
University studies: M.Sc. of College of Computer and Information Engineering from Henan Normal University in China.
Scientific interest: Algorithms, Linear systems
Publications: more than 20 papers published in various journals.
Experience: teaching experience of 10 years, 4 scientific research projects.